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A simple model for the approach of entropy to thermodynamic equilibrium

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Abstract. We consider a simple dynamical system in three different ways, demonstrating that dynamic entropy behaviour can be radically different depending on the perspective. Namely, the Boltzmann–Gibbs entropy of the entire (invertible) system may be constant, increasing or decreasing as a function of time. However, by taking a trace of an invertible dynamical system we may either obtain a system in which the entropy is continuously decreasing or an exact (non-invertible) factor may be obtained which shows a global evolution of entropy to a unique equilibrium.

1. Introduction

For over a century the question of how microscopic reversibility could be reconciled with macroscopic irreversibility has intrigued scientists and generated various attempts to solve the apparent incompatibility of these two properties. The problem was recognized early in the work of Boltzmann and Clausius who attempted to find a dynamical foundation for thermodynamics. Their solution was the *Stosszahlansatz* (molecular-chaos postulate) which even they recognized as being completely *ad hoc*.

In their work, and many subsequent attempts, the dynamical foundation for the investigation of the problem was always taken to be Hamiltonian in form. Hamiltonian systems are intrinsically invertible and, from a dynamic perspective, can be at most ergodic or mixing. However, it is now known that another dynamic property—*exactness*—is both necessary and sufficient for the entropy of a system to globally evolve to a unique state of thermodynamic equilibrium [1, 2]. Exactness is a property that may only be found in non-invertible systems, and thus Hamiltonian systems are automatically excluded as likely dynamical candidates for a foundation of thermodynamics.

An alternative, which has received considerable attention, is coarse graining [1]—a process whereby dynamic information is available with only a certain degree of precision. Although a combination of coarse graining and invertible dynamics is capable of inducing entropy evolution to an equilibrium state, it is incapable of singling out a unique direction of time since this entropy evolution after coarse graining is independent of time reversal. Furthermore, the entropy convergence rate after coarse graining is inversely proportional to the measurement precision, which is in contradiction to all of our usual notions concerning irreversibility.

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Here we adopt a completely novel approach to this old and important question. Using a toy system to illustrate our approach, we consider a two-dimensional system with invertible dynamics operating in a finite phase space. The dynamics are parametrized by a single number c.

We consider a map $S_c : W \to W$, where $W = [0, 1] \times [0, 1]$, to examine the different limiting behaviour of densities and entropy when different perspectives relative to S_c are adopted. Although we have chosen a quite specific form for S_c, we believe our considerations serve as a paradigm for interpreting the longstanding discrepancy between entropy behaviour at the macroscopic level and dynamic properties at the microscopic level.

The specific discrete time map that we have chosen for study is given by

$$S_c(x, y) = (T_c(x), U_c(x, y)) \tag{1}$$

wherein the parametrized maps $T_c, U_c : [0, 1] \to [0, 1]$ are given by

$$T_c(x) = \begin{cases} \dfrac{1-c}{c}x + c & 0 \leqslant x \leqslant c \\ \dfrac{1}{1-c} - \dfrac{1}{1-c}x & c \leqslant x \leqslant 1 \end{cases} \tag{2}$$

(the Mori map, see the left-hand portion of figure 1), and

$$U_c(x, y) = \begin{cases} \alpha y & 0 \leqslant x \leqslant c \\ \alpha + (1-\alpha)y & c < x \leqslant 1 \end{cases} \tag{3}$$

respectively (see figure 1 right), where α and c in (2) and (3) satisfy $\alpha, c \in (0, 1)$.

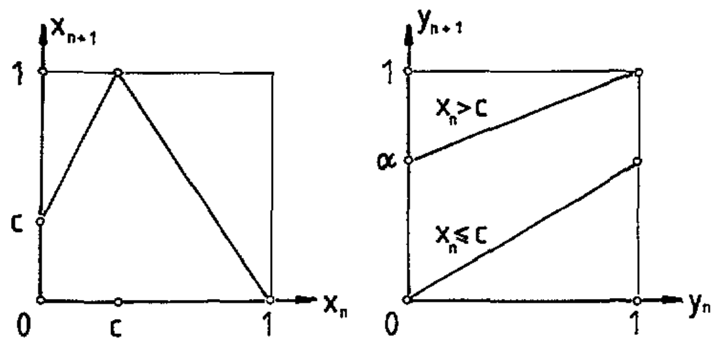


Figure 1. Components of the map S_c defined by (1): left, the map $x_{n+1} = T_c(x_n)$; right, the map $y_{n+1} = U_c(x_n, y_n)$.

Since our goal in this paper is to study the dynamics of entropy evolution from different perspectives, we focus our attention on the *evolution of densities* under the action of S_c and the corresponding evolution of entropies.

The outline of this paper is as follows. In section 2 we introduce some basic concepts and definitions, illustrating these with the properties of the familiar Baker transformation. In section 3 we study the entropy behaviour of the full system S_c as well as the behaviour of the subsystems T_c and U_c. We show that the entropy of the full system S_c may increase, decrease or remain constant as a function of time. However, if we know only the dynamics of T_c we conclude that the entropy always approaches an equilibrium entropy, while knowledge of U_c alone leads us to the conclusion that the entropy is continuously decreasing. We conclude with a discussion of the physical implications of these different ways of examining dynamics and the consequences for entropy in section 4.

2. Tools

Before turning to our central points, we first introduce some concepts, definitions, and results that will be essential [1, 2].

2.1. Dynamics, densities and density evolution

Consider a system operating in a phase space W. On this phase space the temporal evolution of our system is described by a dynamical law S_t that maps points w in the phase space W into new points, i.e. $S_t : W \to W$, as time t changes. In general, W may be a d-dimensional phase space, either finite or not, and therefore w is a d-dimensional vector. Time, t, is discrete and integer valued, $t \in Z = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.

For example, we could consider a system with dynamics described by the Baker transformation

$$S(x, y) = \begin{cases} (2x, \frac{1}{2}y) & 0 \leqslant x \leqslant \frac{1}{2} \\ (2x-1, \frac{1}{2}y + \frac{1}{2}) & \frac{1}{2} < x \leqslant 1 \end{cases} \tag{4}$$

which maps $W = [0, 1] \times [0, 1]$ into itself. We will use the Baker transformation to illustrate the concepts of this section, and lay the foundation for our investigations in subsequent sections.

Two types of dynamics will be important in our considerations. First we introduce the concept of a dynamical system $\{S_t\}_{t \in Z}$ on a phase space W, which is simply any group of transformations $S_t : W \to W$ satisfying: (i) $S_0(w) = w$; and (ii) $S_t(S_{t'}(w)) = S_{t+t'}(w)$ for $t, t' \in Z$. Dynamical systems are invertible since they may be run either forward or backward in time. Other than Hamiltonian dynamics, a good example of an invertible system is given by the Baker transformation since

$$S^{-1}(x, y) = \begin{cases} (\frac{1}{2}x, 2y) & 0 \leqslant y \leqslant \frac{1}{2} \\ (\frac{1}{2}x + \frac{1}{2}, 2y - 1) & \frac{1}{2} < y \leqslant 1. \end{cases} \tag{5}$$

The second type of dynamics, that is important to distinguish, is that of semidynamical systems $\{S_t\}_{t \in N}$, which are any semigroup of transformations $S_t : W \to W$, i.e. (i) $S_0(w) = w$ and (ii) $S_t(S_{t'}(w)) = S_{t+t'}(w)$ for $t, t' \in N = \{0, 1, 2, \ldots\}$. In contrast to dynamical systems, semidynamical systems are non-invertible and may not be run backward in time in an unambiguous fashion. A good example is given if we consider only the x component of the Baker transformation

$$T(x) = \begin{cases} 2x & 0 \leqslant x \leqslant \frac{1}{2} \\ 2x - 1 & \frac{1}{2} \leqslant x \leqslant 1 \end{cases} \tag{6}$$

which can be written in the alternate form $T(x) = 2x \bmod 1$. The dynamics described by (6) are often referred to as the *dyadic map*. It is obvious that T is non-invertible since for any given value of T, there are two possible values of x that could have produced it.

The concept of a factor of a transformation can be understood with the aid of a diagram.

$$\begin{array}{ccc} W & \xrightarrow{S_t} & W \\ {\scriptstyle F}\big\downarrow & & \big\downarrow{\scriptstyle F} \\ X & \xrightarrow[T_t]{} & X \end{array}$$

Let W and X be two different phase spaces. If there is a transformation $F : W \to X$ such that $T_t \circ F = F \circ S_t$ (so the diagram commutes), then T_t is called a factor of S_t. A trajectory of the factor T_t is called a trace of the system S_t.

Going back to our example of the Baker transformation, the transformation T for the x component is a factor of the Baker transformation.

Since we are considering the temporal evolution of densities under the action of S, we examine the way in which the dynamics alter densities. If f is an L^1 function in the space W, i.e. if $\int_W |f(w)|\,\mathrm{d}w < \infty$, then f is a density if $f \in D$ where $D = \{f \in L^1 : f \geqslant 0, \|f\| = 1\}$ denotes the set of all densities. (As usual, $\|f\|$ denotes the L^1 norm $\|f\| = \int_W |f(w)|\,\mathrm{d}w$.) The examination of the evolution of densities by system dynamics is equivalent to examining the behaviour of an infinite number of trajectories. Given a density f then the f-measure $\mu_f(A)$ of the set A in the phase space W is defined by $\mu_f(A) = \int_A f(w)\,\mathrm{d}w$ and f is called the density of the measure μ_f. The usual Lebesgue measure of a set A is denoted by $\mu_{\mathrm{L}}(A)$, and the density of the Lebesgue measure is the uniform density, $f_{\mathrm{L}}(w) = 1/\mu_{\mathrm{L}}(W)$ for all points w in the phase space W. We always write $\mu_{\mathrm{L}}(\mathrm{d}w) = \mathrm{d}w$. Using the notion of an indicator function defined by $1_A(w) = 1$ if $w \in A$ and $1_A(w) = 0$ otherwise, we can write the density of the Lebesgue measure of a set A as $f_{\mathrm{L}}(w) = 1_A(w)/\mu_{\mathrm{L}}(W)$.

Although it is clear from (4) how successive temporal points $(x, y) \in W$ are computed to form the trajectory $\{x_t, y_t\}_{t=0}^{\infty}$, we must introduce an analogous concept for how densities evolve. Any linear operator $P^t : L^1 \to L^1$ that satisfies: (i) $P^t f \geqslant 0$; and (ii) $\|P^t f\| = \|f\|$ for all $t \in Z$ or N and $f \geqslant 0$, $f \in L^1$ is called a Markov operator. If we restrict ourselves to only considering densities f, then any operator P which when acting on a density again yields a density is a Markov operator. Any density f_* that satisfies $P^t f_* = f_*$ for all t is said to be a stationary density of the Markov operator P. In analogy with the definitions of dynamical and semidynamical systems in the last section, we may introduce invertible and non-invertible Markov operators. Given a Markov operator P^t, then P^t is an invertible Markov operator if: (i) $P^0 f = f$; and (ii) $P^t(P^{t'} f) = P^{t+t'} f$ for all $t, t' \in Z$. Clearly, allowing $t, t' \in Z$ is the origin of the invertibility. However, if property (ii) of an invertible Markov operator is replaced by (ii′), $P^t(P^{t'} f) = P^{t+t'} f$ for all $t, t' \in N$, then P^t is a non-invertible Markov operator.

A transformation S_t is said to be measurable if $S_t^{-1}(A) \in W$ for all $A \in W$. Furthermore, given a density f_* and associated measure μ_*, a measurable transformation S_t is non-singular if $\mu_*(S_t^{-1}(A)) = 0$ for all sets A such that $\mu_*(A) = 0$. If S_t is a non-singular transformation, then the unique Markov operator $P^t : L^1 \to L^1$ defined by

$$\int_A P^t f(w)\,\mathrm{d}w = \int_{S_t^{-1}(A)} f(w)\,\mathrm{d}w$$

is called the Frobenius–Perron operator corresponding to S. The Frobenius–Perron operator P^t describes the evolution of densities under the action of a dynamics S. The equation defining the Frobenius–Perron operator has a simple intuitive interpretation. Start with an initial density f and integrate this over a set B that will evolve into the set A under the action of the transformation S_t. However, the set B is $S_t^{-1}(A)$. This integrated quantity must be equal, since S_t is non-singular, to the integral over the set A of the density obtained after one application of S_t to f. This final density is $P^t f$.

Given a density f and associated measure μ_f, then a measurable transformation S_t is said to be f-measure preserving if $\mu_f(S_t^{-1}(A)) = \mu_f(A)$ for all sets A. Measure-preserving transformations are necessarily non-singular. Since the concept of measure preservation is not only dependent on the transformation but also on the measure, we alternately say that the measure μ_f is invariant with respect to the transformation S_t if S_t is f measure preserving.

It is easily shown that the Baker transformation preserves the Lebesgue measure on $W = [0, 1] \times [0, 1]$ since an expansion in the x direction by a factor of two is always compensated for by the corresponding contraction factor of $1/2$ in the y direction. The Frobenius–Perron operator corresponding to the Baker transformation is given by

$$P_S f(x, y) = \begin{cases} f(\frac{1}{2}x, 2y) & 0 \leqslant y \leqslant \frac{1}{2} \\ f(\frac{1}{2}x, 2y - 1) & \frac{1}{2} < y \leqslant 1. \end{cases} \tag{7}$$

Clearly, $P_S 1_W(x, y) = 1_W(x, y)$ illustrating that the uniform density of the Lebesgue measure $\mu_L([0, 1] \times [0, 1])$ is a stationary density of P_S.

2.2. *Ergodicity, mixing and exactness*

We next turn to a consideration of the dynamical properties of maps $S : W \to W$ as manifested through the behaviour of sequences of densities $\{P^t f\}$ where P is the Frobenius–Perron operator corresponding to S with stationary density f_*.

First, a non-singular transformation S_t is said to be f_* ergodic if $\{P^t f\}$ is Cesàro convergent to f_* for all densities f, i.e., if $\lim_{t\to\infty}(1/t)\sum_{k=0}^{t-1}\langle P^k f, g\rangle = \langle f_*, g\rangle$. (Here, the scalar product of two functions is denoted in the usual way: $\langle f, g\rangle = \int_X f(x)g(x)\,dx$ where $f \in L^1$ and $g \in L^\infty$.) Ergodicity is completely equivalent to the existence of a *unique* stationary density f_*. Secondly, let S_t be an f_* measure-preserving transformation operating on a finite normalized space. Then S_t is called f_* mixing if

$$\lim_{t\to\infty}\langle P^t f, g\rangle = \langle f_*, g\rangle$$

i.e. the sequence $\{P^t f\}$ is weakly convergent to the density f_* for all initial densities f.

Thirdly, if S_t is an f_* measure-preserving transformation operating on a normalized phase space W, then S_t is said to be f_* exact if

$$\lim_{t\to\infty}\|P^t f - f_*\| = 0 \qquad \text{for all } f \in D$$

i.e. $\{P^t f\}$ is strongly convergent to f_* for all initial densities f. Exactness is completely equivalent to $\lim_{t\to\infty}\mu_*(S_t(A)) = 1$ for all sets A of non-zero measure. It is important to note that *systems with invertible dynamics can never be exact.* Exactness implies mixing which, in turn, implies ergodicity. If a given dynamics is ergodic, mixing or exact we will also use the same adjective for the corresponding Frobenius–Perron operator. Intermediate between mixing and exactness is a fourth type of dynamics known as a K automorphism, but for all practical purposes we may take this as equivalent to mixing.

The Baker transformation (4) is not only ergodic, it is also mixing (actually, it is more than mixing since it is a K-automorphism, but this will not concern us here). Because of its invertibility, it cannot be exact. However, if we consider the factor $T(x)$ of the Baker system, it is well known that the dyadic map has a unique stationary density $f_*(x) = 1_{[0,1]}(x)$ for $0 \leqslant x \leqslant 1$ and, further, is exact.

2.3. Entropy

A central consideration in this paper is the behaviour of the entropy of a density. We first define the Boltzmann–Gibbs entropy of a density f by

$$H(f) = -\int_W f(w) \log f(w)\, \mathrm{d}w \tag{8}$$

in keeping with the introduction of entropy in the seminal work of Boltzmann and Gibbs.

The Boltzmann–Gibbs entropy of the Baker transformation is given by

$$H(P_S f) = -\int_0^{1/2}\int_0^1 f(\tfrac{1}{2}x, 2y) \log f(\tfrac{1}{2}x, 2y)\, \mathrm{d}x\, \mathrm{d}y - \int_{1/2}^{1}\int_0^1 f(\tfrac{1}{2}x, 2y-1) \log f(\tfrac{1}{2}x, 2y-1)\, \mathrm{d}x\, \mathrm{d}y. \tag{9}$$

A change of variables on the right-hand side gives $H(P_S f) \equiv H(f)$, so $H(P_S^t f) = H(f)$ for all times t, illustrating that the Boltzmann–Gibbs entropy of an invertible measure-preserving system is always constant in time. Furthermore, since the Baker transformation is ergodic we know that the stationary density $f_*(x, y) = 1$ is unique. It is straightforward to show that $H(1_W) \equiv 0$ so the Boltzmann–Gibbs entropy of the full Baker transformation will, in general, not be equal to the entropy of the stationary density.

Continuing with the Baker system, we suppose that although the dynamics (4) continue to operate we are ignorant of the existence of the variable y and are only able to measure values of the x variable (i.e. we are only able to examine a trace of S). Thus, through our measurements we are monitoring the behaviour of the exact dyadic map (6) in complete ignorance of the existence of the concomitant dynamics of y.

The entropic implications of the dynamic property of exactness are contained in the following result, which also offers an interesting commentary about the second law of thermodynamics.

Theorem 1. Let P^t be a Markov operator operating in a phase space W. Then the entropy of $P^t f$ goes to its equilibrium value $H(f_*)$ as $t \to \infty$

$$\lim_{t\to\infty} H(P^t f) = H(f_*) \qquad \text{for all } f \in D$$

if and only if P^t is f_* exact.

Remark. Theorem 1 tells us that in only examining the trace of the factor T (given by (6)) of the full dynamics (4), i.e. by only observing x, we will conclude that the entropy of the observed system uniquely converges to the entropy of the stationary density! This is clearly in contrast to our conclusions when we observed the entropy behaviour under the action of the full dynamics S.

The proof of this theorem is simple given the notion of conditional entropy, and a lemma.

If f and g are two densities such that $\operatorname{supp} f \subset \operatorname{supp} g$ ($\operatorname{supp} f$ denotes the support of f), then the conditional entropy of the density f with respect to the density g is

$$H_c(f|g) = -\int_W f(w) \log\left[\frac{f(w)}{g(w)}\right] \mathrm{d}w.$$

The conditional entropy, a generalization of the Boltzmann–Gibbs entropy, is always defined and $H_c(f|g)$ measures the deviation between the two densities f and g. The following lemma deals with the global convergence properties of H_c.

Lemma 2. (Mackey [1], theorem 7.7.) Let P^t be a Markov operator operating in a phase space W with stationary density f_*. Then the conditional entropy of $P^t f$ with respect to f_* goes to zero

$$\lim_{t\to\infty} H_c(P^t f | f_*) = 0 \qquad \text{for all } f \in D$$

if and only if P^t is f_* exact.

Proof of theorem 1. The proof follows directly from lemma 2 and the definition of f_* exactness when we rewrite the conditional entropy of $P^t f$ with respect to f_* in the form:

$$H_c(P^t f | f_*) = H(P^t f) - H(f_*) + \int_W [P^t f(w) - f_*(w)] \log f_*(w)\, dw. \quad \square$$

In a general framework, theorem 1 is remarkable for two reasons:

(i) It sets forth necessary and sufficient *dynamic* criteria for the second law of thermodynamics; and

(ii) Since all exact systems are non-invertible, and all microscopic dynamical equations of motion in physics are invertible, it highlights a clear problem that will be met in any attempt to reconcile macroscopic thermodynamic behaviour with microscopic dynamics as currently formulated.

Continuing this examination of the entropy behaviour of exact transformations and the question of factors, we close this section with the following theorem.

Theorem 3. (Rochlin [3]) Every f_*-exact transformation is the factor of a K-automorphism.

The transformation F involved in our discussion of factors and traces is precisely what Prigogine and co-workers [4, 5] refer to as a projection operator in their work on the generation of irreversible behaviour from reversible dynamics. The Rochlin theorem 3 serves as the analytic link in their work between K-automorphisms and exact transformations.

Since K-automorphisms are invertible and measure preserving, their entropy is forever fixed at its initial value [1]. On the other hand, by theorem 1 we know that the entropy of an exact transformation, which by theorem 3 is the factor of a K-automorphism, converges to its equilibrium value irrespective of the initial density with which the system was prepared.

3. Three views of dynamics: three entropy behaviours

With the introductory and illustrative material of the previous section, we now turn to our main subject—the investigation of the system (1)–(3), parametrized by $c \in (0, 1)$. Thus the full system S_c is a combination of a tent-like map T_c (equation (2)) as considered by Mori and co-workers [6], and a slight generalization (U_c, equation (3)) of the y portion of the Baker transformation.

3.1. Entropy of S_c

Like the Baker transformation, the map S_c is invertible and has an inverse given by

$$S_c^{-1}(x,y) = \begin{cases} \left(\dfrac{c}{1-c}(x-c), \dfrac{y}{\alpha}\right) & 0 \leqslant y \leqslant \alpha \\ \left(1-(1-c)x, \dfrac{y-\alpha}{1-\alpha}\right) & \alpha < y \leqslant 1. \end{cases} \tag{10}$$

From this relation, it follows that the Frobenius–Perron operator P_{S_c} corresponding to S_c is

$$\begin{aligned} P_{S_c} f(x,y) = {} & \frac{c}{\alpha(1-c)} f\left(\frac{c}{1-c}(x-c), \frac{y}{\alpha}\right) 1_{[c,1]}(x) 1_{[0,\alpha]}(y) \\ & + \frac{1-c}{1-\alpha} f\left(1-(1-c)x, \frac{y-\alpha}{1-\alpha}\right) 1_{[\alpha,1]}(y). \end{aligned} \tag{11}$$

Although S_c is invertible, it is not measure preserving, and thus the Boltzmann–Gibbs entropy of $P_{S_c} f$ is, in general, not constant with respect to time. Indeed, a straightforward calculation gives

$$\begin{aligned} H(P_{S_c} f) &= -\int_0^1 \int_0^1 P_{S_c} f(x,y) \log P_{S_c} f(x,y)\, dx\, dy \\ &= H(f) - \left[\log \frac{c}{\alpha(1-c)}\right] \int_0^c \bar{f}(x)\, dx - \left[\log \frac{1-c}{1-\alpha}\right] \int_c^1 \bar{f}(x)\, dx \end{aligned} \tag{12}$$

where

$$H(f) = -\int_0^1 \int_0^1 f(x,y) \log f(x,y)\, dx\, dy$$

and

$$\bar{f}(x) = \int_0^1 f(x,y)\, dy.$$

An examination of the coefficients

$$\log \frac{c}{\alpha(1-c)} \qquad \text{and} \qquad \log \frac{1-c}{1-\alpha}$$

in (12) shows that it is impossible for them to be simultaneously negative. There are two possibilities:

(i) If they have different signs, then the entropy $H(P_{S_c} f)$ can be equal to, less than or greater than $H(f)$ depending on the values of the integrals over the reduced density $\bar{f}(x)$ of the factor T_c.

(ii) Alternatively, if both are positive (which occurs for $c < \alpha < c/(1-c)$) then the full dynamics S_c are contracting and the entropy is decreasing, $H(P_{S_c} f) < H(f)$.

3.2. Entropy evolution of the factor T_c

The Frobenius–Perron operator corresponding to T_c is given by [6]

$$P_{S_c} f(x) = \frac{c}{1-c} f\left(\frac{c}{1-c}(x-c)\right) 1_{[c,1]}(x) + (1-c) f(1-(1-c)x) \tag{13}$$

and it has a parametrized stationary density

$$f_*(x; c) = \frac{1}{1+c} 1_{[0,c]}(x) + \frac{1}{1-c^2} 1_{(c,1]}(x) \qquad c \in (0, 1). \tag{14}$$

The map T_c is a factor of the full two dimensional dynamics S_c since the following commutative relation holds

$$\begin{array}{ccc} f(x,y) & \xrightarrow{P_{S_c}} & P_{S_c} f(x,y) \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F} \\ f(x) & \xrightarrow[P_{T_c}]{} & P_{T_c} f(x) \end{array}$$

so

$$P_{T_c} \int f(x, y)\,\mathrm{d}y = \int P_{S_c} f(x, y)\,\mathrm{d}y$$

by identifying the factor operator with $Ff(x, y) = \int f(x, y)\,\mathrm{d}y$. This can be shown in a straightforward fashion using the explicit expressions for P_{T_c} and P_{S_c}.

Furthermore, the map T_c is f_* exact where f_* is the stationary density given in (14). To prove this requires a minor digression.

We first define a non-trivial lower bound function $h \in L^1$ for a Markov operator P as any function $h \geqslant 0$ with $\|h\| > 0$ such that $\lim_{t\to\infty} P^t f \geqslant h$ for all initial densities f. Then the following lemma is useful.

Lemma 4. (Mackey [1], Theorem 7.6.) A Markov operator P is f_* exact if and only if there exists a non-trivial lower bound function h for P.

We can use this result to prove the exactness of the Mori map. Namely

Theorem 5. The Mori map (2) is f_* exact for all $c \in (0, 1)$.

Proof. Pick an arbitrary set $A \subset [0, 1]$ with non-zero Lebesgue measure $\mu_L(A) > 0$. Note that for every two iterations of the Mori map, we have an expansion of the measure by a factor of at least $1/c$ so in a finite number of steps

$$\left(\frac{1}{c}\right)^t \mu_L(A) = \mu_L(S^{2t}(A)) \geqslant 1$$

and for all $t > t_0(f)$, where

$$t_0(f) \geqslant \frac{\log(1/(\mu_L(a)))}{\log(1/c))}$$

we have that $\operatorname{supp} P^t_{T_c} f = [0, 1]$.

Now let $h = 1_{[0,1]}(x) \inf_x[\lim_{t\to\infty} P^t_{T_c} f(x)])$. By our above arguments, h is a non-trivial lower bound function for P_{T_c} and T_c is f_* exact. □

As a consequence of the exactness of the factor T_c of S_c, we know from theorem 1 that the entropy $H(P^t_{T_c} f)$ will approach its equilibrium value of

$$H_*(c) \equiv H(f_*) = \log(1+c) + \frac{1}{1+c}\log(1-c) \tag{15}$$

as $t \to \infty$.

Comparison of this result with theorem 3 offers an interesting parallel. Namely, in our situation we have a non-measure-preserving invertible map S_c instead of the Baker transformation which is a measure-preserving K automorphism. Nevertheless, in the situation we consider here taking a trace (of S_c) to give an exact factor (T_c) gives rise to a system whose entropy approaches the equilibrium value.

We will see in the next section that this global approach of $H(P^t_{T_c} f)$ to the entropy of f_* is accompanied by corresponding changes in the entropy of $P^t_{U_c} f$.

3.3. Entropy evolution and U_c

We next examine the temporal evolution of densities under the action of U_c and the corresponding entropy. We do this for two different cases.

(i) In the first, we consider a situation in which the evolution of x can be described exactly, i.e. by a trajectory which is given by the iteration of the map T_c of equation (2). Then the evolution of y is described in terms of a density $\hat{f}(y)$ which evolves under the influence of the trajectory $x(t)$.

(ii) In the second, the evolution of both variables (x and y) is described by a density $f(x, y)$ as in section 3.1, but then by integration over x a reduced or traced density $\tilde{f}(y)$ is considered.

3.4. Case (i)

In this case, the density $\hat{f}$ translates to $P_{U_c}\hat{f}$ according to

$$P_{U_c}\hat{f}(y) = \begin{cases} \dfrac{1}{\alpha}\hat{f}\left(\dfrac{y}{\alpha}\right) & 0 \leqslant x \leqslant c \\ \dfrac{1}{1-\alpha}\hat{f}\left(\dfrac{y-\alpha}{1-\alpha}\right) & c < x \leqslant 1. \end{cases} \tag{16}$$

Notice that the expression for $P_{U_c}\hat{f}$ is like a Frobenius–Perron operator, yet different for each time step because of its dependence on x. Since from equation (16) the action of P_{U_c} is always contracting with $\alpha \in (0, 1)$, we know that the entropy $H(P^t_{U_c}\hat{f})$ is a strictly decreasing function of increasing time. Since T_c is f_* exact, the evolution of y under the action of U_c can be alternatively interpreted as due to the action of a random map [7].

As this behaviour is totally independent of the trajectory $\{x_t\}_{t=0}^{\infty}$ we should expect that it will also hold for an ensemble of trajectories described by the combined density $f(x, y)$. This expectation is confirmed by the following calculation, corresponding to the second case listed above.

3.5. Case (ii)

Consider a density $f(x, y)$ of the combined system whose evolution is governed by the Frobenius–Perron operator (11) of section 3.1. Then a 'traced' density $\tilde{f}(y)$ is defined by

$$\tilde{f}(y) = \int_0^1 f(x, y)\,\mathrm{d}x.$$

Furthermore,

$$P_{U_c}\tilde{f}(y) = \int_0^1 P_{S_c}f(x,y)\,\mathrm{d}x = 1_{[0,\alpha]}(y)\int_c^1 \frac{c}{\alpha(1-c)} f\left(\frac{c}{1-c}(x-c), \frac{y}{\alpha}\right)\mathrm{d}x$$
$$+\,1_{(\alpha,1]}(y)\int_0^1 \frac{1-c}{1-\alpha} f\left(1-(1-c)x, \frac{y-\alpha}{1-\alpha}\right)\mathrm{d}x$$
$$= 1_{[0,\alpha]}(y)\int_0^c \frac{1}{\alpha} f\left(x, \frac{y}{\alpha}\right)\mathrm{d}x + 1_{(\alpha,1]}(y)\int_c^1 \frac{1}{1-\alpha} f\left(x, \frac{y-\alpha}{1-\alpha}\right)\mathrm{d}x.$$

We will first prove that the weak inequality $H(P_{U_c}\tilde{f}) \leqslant H(\tilde{f})$ holds, and then show that, on average, the stronger relation $H(P_{U_c}\tilde{f}) < H(\tilde{f})$ holds on every second iteration.

To show that the weak inequality $H(P_{U_c}\tilde{f}) \leqslant H(\tilde{f})$ is always valid, note that the entropy of $P_{U_c}\tilde{f}$ can be written

$$H(P_{U_c}\tilde{f}) = -\int_0^\alpha \left\{\left[\int_0^c \frac{1}{\alpha} f\left(x, \frac{y}{\alpha}\right)\mathrm{d}x\right] \log\left[\int_0^c \frac{1}{\alpha} f\left(x, \frac{y}{\alpha}\right)\mathrm{d}x\right]\right\}\mathrm{d}y$$
$$-\int_\alpha^1 \left\{\left[\int_c^1 \frac{1}{1-\alpha} f\left(x, \frac{y-\alpha}{1-\alpha}\right)\mathrm{d}x\right] \log\left[\int_c^1 \frac{1}{1-\alpha} f\left(x, \frac{y-\alpha}{1-\alpha}\right)\mathrm{d}x\right]\right\}\mathrm{d}y$$
$$= -\int_0^1 \left\{\left[\int_0^c f(x,y)\,\mathrm{d}x\right] \log\left[\frac{1}{\alpha}\int_0^c f(x,y)\,\mathrm{d}x\right]\right\}\mathrm{d}y$$
$$-\int_0^1 \left\{\left[\int_c^1 f(x,y)\,\mathrm{d}x\right] \log\left[\frac{1}{1-\alpha}\int_c^1 f(x,y)\,\mathrm{d}x\right]\right\}\mathrm{d}y.$$

If we define

$$\tilde{f}_1(y) = \int_0^c f(x,y)\,\mathrm{d}x \qquad \text{and} \qquad \tilde{f}_2(y) = \int_c^1 f(x,y)\,\mathrm{d}x$$

so $\tilde{f} = \tilde{f}_1 + \tilde{f}_2$, then

$$H(P_{U_c}\tilde{f}) = -\int_0^1 \left\{\tilde{f}_1(y)\log\frac{\tilde{f}_1(y)}{\alpha} + \tilde{f}_2(y)\log\frac{\tilde{f}_2(y)}{1-\alpha}\right\}\mathrm{d}y$$
$$= -\int_0^1 \left\{p_1(y)\log\frac{p_1(y)\tilde{f}(y)}{\alpha} + p_2(y)\log\frac{p_2(y)\tilde{f}(y)}{1-\alpha}\right\}\tilde{f}(y)\,\mathrm{d}y$$
$$= \int_0^1 \left\{p_1(y)\log\frac{\alpha}{p_1(y)\tilde{f}(y)} + p_2(y)\log\frac{1-\alpha}{p_2(y)\tilde{f}(y)}\right\}\tilde{f}(y)\,\mathrm{d}y$$

where $p_i(y) = \tilde{f}_i(y)/\tilde{f}(y)$, for $i = 1, 2$ and $y \in \operatorname{supp}\tilde{f}$ with $\tilde{f}(y) \neq 0$, so $p_1 + p_2 \equiv 1$.

Using the relation

$$p_1 \log a + p_2 \log b \leqslant \log(p_1 a + p_2 b)$$

we have immediately

$$H(P_{U_c}\tilde{f}) \leqslant H(\tilde{f}).$$

In point of fact, we can sharpen this relation between successive entropies considerably, since in general $H(P_{U_c}\tilde{f}) < H(\tilde{f})$ whenever $p_1 \log a + p_2 \log b < \log(p_1 a + p_2 b)$. The only cases for which $p_1 \log a + p_2 \log b = \log(p_1 a + p_2 b)$ are: (i) $p_1 = 0$ or $p_2 = 0$; or (ii) $a = b$. We consider each in turn.

(i) If $p_1(y) \equiv 0$ then

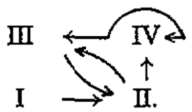
$$H(P_{U_c}\tilde{f}) = -\int_0^1 \tilde{f}_2(y) \log \tilde{f}_2(y)\, \mathrm{d}y - \log \frac{1}{1-\alpha} < H(\tilde{f}).$$

The same conclusion holds in the event that $p_2(y) \equiv 0$; or if $p_1(y) \equiv 0$ for $y \in A_1$ and $p_2(y) \equiv 0$ for $y \in A_2$ with $A_1, A_2 \subset [0, 1]$ and $A_1 \cup A_2 = \operatorname{supp} \tilde{f}$. Thus, again $H(P_{U_c}\tilde{f}) < H(\tilde{f})$.

(ii) From the above considerations it is obvious that a necessary condition for the equality $H(P_{U_c}\tilde{f}) = H(\tilde{f})$ to hold is $a = b$ or

$$\frac{1}{\alpha}\tilde{f}_1(y) = \frac{1}{1-\alpha}\tilde{f}_2(y) \qquad \text{for all } y.$$

Consider the phase space of S_c, consisting of the unit square, divided into four regions as illustrated in figure 2. A straightforward consideration of the map (1) shows that under the action of S_c the flows between these four regions is given by



Since there is no input to region I it follows that, in general (after transients), $H(P_{U_c}\tilde{f}) = H(\tilde{f})$ if and only if supp $\tilde{f}_2$ is contained in region IV. Because of the flows between the four regions, it may be the case that on one iteration $H(P_{U_c}\tilde{f}) = H(\tilde{f})$, but for the next $H(P^2_{U_c}\tilde{f}) < H(P_{U_c}\tilde{f})$. Thus we conclude that, on an average encompassing two or more iterations, the entropy $H(P^t_{U_c}\tilde{f})$ is strictly decreasing.

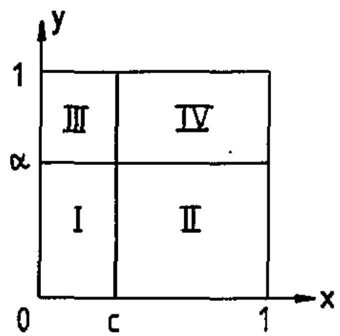


Figure 2. Four regions of the phase space W.

4. Conclusions

In this paper we have considered a simple system in three different ways, demonstrating that dynamic entropy behaviour can be radically different depending on the perspective. Namely, the entropy of the entire (invertible) system S_c may be constant, increasing or decreasing as a function of time. However, by taking a trace of an invertible dynamical system we may either obtain a system (U_c) in which the entropy is continuously decreasing, or an exact (non-invertible) factor (T_c) may be obtained which shows a global evolution of entropy to a unique equilibrium.

Even though the system we consider is extremely simple, the fact that it is capable of displaying such a wide range of behaviour normally associated with much more complicated systems leads us to speculate whether it offers an important paradigm for extending our understanding of equilibrium and non-equilibrium thermodynamic behaviour.

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